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Supersymmetry and the generalized Lichnerowicz formula

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Abstract. A classical result in differential geometry due to Lichnerowicz [8] is concerned with the decomposition of the square of Dirac operators defined by Clifford connections on a Clifford module $\mathcal{E}$ over a Riemannian manifold M. Recently, this formula has been generalized to arbitrary Dirac operators [2]. In this paper we prove a supersymmetric version of the generalized Lichnerowicz formula, motivated by the fact that there is a one-to-one correspondence between Clifford superconnections and Dirac operators. We extend this result to obtain a simple formula for the supercurvature of a generalized Bismut superconnection. This might be seen as a first step to prove the local index theorem also for families of arbitrary Dirac operators.

Keywords: *supersymmetry, generalized Lichnerowicz formula, generalized Bismut superconnections, local family index theorem*

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1. Introduction

A classical result in differential geometry due to Lichnerowicz [8] is concerned with the decomposition of the square of Dirac operators defined by Clifford connections on a Clifford module $\mathcal{E}$ over a Riemannian manifold M. More precisely, if M is even dimensional, $\nabla^{\mathcal{E}}\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ is such a connection and $D_{\nabla^{\mathcal{E}}} := c \circ \nabla^{\mathcal{E}}$ denotes the corresponding Dirac operator, Lichnerowicz's formula states

$$D^2_{\nabla^{\mathcal{E}}} = \triangle^{\nabla^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}(R^{\mathcal{E}/S}_{\nabla^{\mathcal{E}}}). \tag{1.1}$$

Here $\triangle^{\nabla^{\mathcal{E}}}$ is the connection laplacian associated to $\nabla^{\mathcal{E}}$ and the endomorphism part is given by the scalar curvature r_M of M and the image $\mathbf{c}(R^{\mathcal{E}/S}_{\nabla^{\mathcal{E}}})$ of the twisting curvature $R^{\mathcal{E}/S}_{\nabla^{\mathcal{E}}} \in \Omega^2(M, \mathrm{End}_{C(M)}(\mathcal{E}))$ of the Clifford connection $\nabla^{\mathcal{E}}$ with respect to the quantisation map $\mathbf{c}\colon \Lambda^*T^*M \to C(M)$, cf. [5]. This formula provides a powerful tool for studying the Dirac operator $D_{\nabla^{\mathcal{E}}}$ and its relation with the geometry of the underlying manifold. For example, if M is a compact spin manifold with positive scalar curvature, Lichnerowiz used (1.1) to show the vanishing of the space of harmonic spinors. Moreover, formula (1.1) assumes a significant rôle in the proof of the local Atiyah-Singer index theorem for such Dirac operators, cf. [6].

Recently, Lichnerowicz's formula (1.1) has been extended to arbitrary Dirac operators: In [2] it was shown that the square $\widetilde{D}^2$ of a Dirac operator $\widetilde{D} := c \circ \widetilde{\nabla}^{\mathcal{E}}$ defined by an arbitrary connection $\widetilde{\nabla}^{\mathcal{E}}\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ on the Clifford module $\mathcal{E}$ decomposes as

$$\widetilde{D}^2 = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \mathbf{c}(R^{\widetilde{\nabla}^{\mathcal{E}}}) + ev_g \widetilde{\nabla}^{T^*M \otimes \mathrm{End}\mathcal{E}} \varpi_{\widetilde{\nabla}^{\mathcal{E}}} + ev_g(\varpi_{\widetilde{\nabla}^{\mathcal{E}}} \cdot \varpi_{\widetilde{\nabla}^{\mathcal{E}}}), \tag{1.2}$$

with $\varpi_{\widetilde{\nabla}^{\mathcal{E}}} := -\frac{1}{2} g_{\nu\kappa} dx^\nu \otimes c(dx^\mu)\big([\widetilde{\nabla}^{\mathcal{E}}_\mu, c(dx^\kappa)] + c(dx^\sigma)\Gamma^\kappa_{\sigma\mu}\big) \in \Omega^1(M, \mathrm{End}\mathcal{E})$ and the connection $\hat{\nabla}^{\mathcal{E}} := \widetilde{\nabla}^{\mathcal{E}} + \varpi_{\widetilde{\nabla}^{\mathcal{E}}}$. Here the last two terms obviously indicate the deviation of the connection $\widetilde{\nabla}^{\mathcal{E}}$ being a Clifford connection. Only the second term in (1.2) is endowed with geometric significance. Of course, if $\widetilde{\nabla}^{\mathcal{E}}$ is a Clifford connection, obviously $\varpi_{\widetilde{\nabla}^{\mathcal{E}}} = 0$ and therefore (1.2) reduces to (1.1).

Using Quillen's theory of superconnections on $\mathbb{Z}_2$-graded vectorbundles nowadays it is well-known that that any Clifford superconnection $\mathbb{A}$ on $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ uniquely determines a Dirac operator $D_{\mathbb{A}}$ due to the following construction

$$D_{\mathbb{A}}\colon \Gamma(\mathcal{E}) \xrightarrow{\mathbb{A}} \Omega^*(M, \mathcal{E}) \xrightarrow{\mathbf{c} \otimes \mathbb{1}_{\mathcal{E}}} \Gamma(C(M) \otimes \mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E}), \tag{1.3}$$

i.e. there is a one-to-one correspondence between Clifford superconnections and Dirac operators, see [5]. In contrast to [2] here we emphasize this approach to Dirac operators. Thus, the first major purpose of this paper consists in proving the supersymmetric version

$$D_{\mathbb{A}}^2 = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big) + \mathbf{c}\big(\bar{\mathbb{A}}\big)^2 - \mathbf{c}\big(\bar{\mathbb{A}}^2\big) + ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big) \tag{1.3}$$

of the intrinsic decomposition formula (1.2) in Theorem 4.2. Here the connection $\hat{\nabla}^{\mathcal{E}} := \mathbb{A}_{[1]} + \beta(\bar{\mathbb{A}})$ is determined by the connection part $\mathbb{A}_{[1]}$ and the connection-free part $\bar{\mathbb{A}} \in \bigoplus_{i \neq 1} \Omega^i(M, \mathrm{End}\,\mathcal{E})$ of the Clifford superconnection $\mathbb{A}$, $\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}$ denotes the twisting supercurvature of $\mathbb{A}$ and $\beta(\bar{\mathbb{A}}) \in \Omega^1(M, \mathrm{End}\,\mathcal{E})$ is defined by $\beta(\bar{\mathbb{A}}) := dx^k \otimes \mathbf{c}\big(i(\partial_k)\bar{\mathbb{A}}\big)$ with respect to a local coordinate frame. Note that Getzler [7] has stated the generalization of Lichnerowicz formula as $D_{\mathbb{A}}^2 = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big) + P(\mathbb{A})$ without specifying the endomorphism $P(\mathbb{A}) \in \Gamma(\mathrm{End}\ \mathcal{E})$ in general. In view of explicit computations, (1.2) resp. (1.3) are more convenient, e.g. in applications to physics (cf. [2], [3]).

Secondly, we extend the above formula (1.3) to a family of Dirac operators $\mathsf{D}_{\mathbb{A}} := \{\ D_{\mathbb{A}} \mid z \in B\ \}$ parametrized by a - not necesarily finite dimensional - manifold B. In order to do so we associate to each family $\mathsf{D}_{\mathbb{A}}$ a superconnection $\nabla^{\mathbb{A}}$ following Bismut's construcion [4]. In the case of $\mathsf{D}_{\mathbb{A}}$ being a family of Dirac operators defined by Clifford connections, $\nabla^{\mathbb{A}}$ reduces to the Bismut superconnection. So we call it 'generalized Bismut superconnection'.

The paper is organized as follows: In the next section we establish our conventions and briefly recall Quillen's superconnection formalism. In section 3 we establish the canonical projection $\mathbf{g}\colon \Omega^*(M, \mathcal{E}) \to \Omega^1(M, \mathrm{End}\ \mathcal{E})$ and show some identities which are crucial in proving our Theorem 4.2 . Although this is basic for the theory of Clifford modules we recognize that there is no presentation available in the literature. In section 4 we prove our main result, the supersymmetric version of the generalized Lichnerowicz formula (1.2) in Theorem 4.2 as we have already mentioned. Finally, in the last section we extend this formula in Theorem 5.2 to the context of families of Dirac operators by generalizing Bismut's construction.

In another paper [1] we will show how to use Theorem 5.2 to compute explicitly the Chern character of a generalized Bismut superconnection which implies the local Atiyah-Singer index theorem for families of arbitrary Dirac operators.

2. The superconnection formalism

In this section we briefly introduce some basic results of Quillen [9] concerning supersymmetry and superconnections. Let $E = E^+ \oplus E^-$ be a finite dimensional $\mathbb{Z}_2$-graded vector bundle over a manifold M. Then the endomorphism bundle $\mathrm{End}E$ is canonicaly graded and - by their total $\mathbb{Z}_2$-gradings - so are the spaces $\Omega^*(M, E)$, $\Omega^*(M, \mathrm{End}E)$ of E-vallued and $\mathrm{End}E$-vallued differential forms, respectively.

A superconnection on E is defined to be an odd parity operator $\mathbb{A}: \Gamma(E) \to \Omega^*(M, E)$ which satisfies Leibniz's rule $\mathbb{A}(fs) = df \otimes s + f\mathbb{A}s$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$. Obviously, $\mathbb{A}$ can be extended to an odd operator of $\Omega^*(M, E)$. Furthermore we can expand $\mathbb{A}$ into a sum $\mathbb{A} = \sum_{i\geq 0} \mathbb{A}_{[i]}$ of operators $\mathbb{A}_{[i]}: \Omega^*(M, E) \to \Omega^{*+i}(M, E)$ such that $\mathbb{A}_{[1]}$ is a connection on E which respects the grading and $\mathbb{A}_{[i]} \in \Omega^i(M, \mathrm{End}E)^-$ for $i \neq 1$. Any superconnection $\mathbb{A}$ acts on the space $\Omega^*(M, \mathrm{End}E)$ by $\mathbb{A}\alpha := [\mathbb{A}, \alpha]$ for $\alpha \in \Omega^*(M, \mathrm{End}E)$. Here $[\,\cdot\,,\,\cdot\,]$ denotes the supercommutator

$$[\alpha, \alpha'] := \alpha\alpha' - (-1)^{|\alpha|\,|\alpha'|}\alpha'\alpha \tag{2.1}$$

where $\alpha, \alpha' \in \Omega^*(M, \mathrm{End}E)$. By definition, the curvature of a superconnection $\mathbb{A}$ is $\mathbb{F}(\mathbb{A}) := \mathbb{A}^2$ which is an even $C^\infty(M)$-linear endomorphism of $\Omega^*(M, E)$, i.e. $\mathbb{F}(\mathbb{A}) \in \Omega^*(M, \mathrm{End}E)^+$. Also the supercurvature $\mathbb{F}(\mathbb{A})$ splits into a sum $\mathbb{F}(\mathbb{A}) = \sum_{i\geq 0} \mathbb{F}(\mathbb{A})_{[i]}$ with $\mathbb{F}(\mathbb{A})_{[i]} \in \Omega^i(M, \mathrm{End}E)^+$ given by

$$\begin{aligned}
\mathbb{F}(\mathbb{A})_{[0]} &= \mathbb{A}_{[0]}^2 \\
\mathbb{F}(\mathbb{A})_{[1]} &= [\mathbb{A}_{[0]}, \mathbb{A}_{[1]}] \\
\mathbb{F}(\mathbb{A})_{[2]} &= [\mathbb{A}_{[0]}, \mathbb{A}_{[2]}] + \mathbb{A}_{[1]}^2 \\
&\;\vdots \\
\mathbb{F}(\mathbb{A})_{[i]} &= [\mathbb{A}_{[0]}, \mathbb{A}_{[i]}] + [\mathbb{A}_{[1]}, \mathbb{A}_{[i-1]}] + \ \ldots \\
&\;\vdots
\end{aligned} \tag{2.2}$$

Now let $\nabla^E := \mathbb{A}_{[1]}$ be the connection- and $\bar{\mathbb{A}} \in \bigoplus_{i\neq 1} \Omega^i(M, \mathrm{End}E)$ be the 'connection-free' part of the superconnection A such that the decomposition $\mathbb{A} = \nabla^E + \bar{\mathbb{A}}$ holds. Because we have $[\nabla^E, \alpha] = d^{\nabla^{End\ E}}\alpha$ for all $\alpha \in \Omega^*(M, \mathrm{End}\ E)$ where

$d^{\nabla^{End\ E}}$ denotes the induced exterior covariant derivative on $\Omega^*(M, \mathrm{End}\ E)$, the above decomposition (2.2) implies the following

LEMMA 2.1. *Let $R^{\nabla^E} \in \Omega^2(M, \mathrm{End}E)$ denote the curvature of the connection part $\nabla^E := \mathbb{A}_{[1]}$ of a superconnection $\mathbb{A} \in \Omega^*(M, \mathrm{End}E)$. Then the supercurvature $\mathbb{F}(\mathbb{A}) \in \Omega^*(M, \mathrm{End}E)^+$ splits into $\mathbb{F}(\mathbb{A}) = R^{\nabla^E} + d^{\nabla^{End\ E}}\bar{\mathbb{A}} + \bar{\mathbb{A}}^2$ with $\bar{\mathbb{A}} \in \bigoplus_{i\neq 1} \Omega^i(M, \mathrm{End}E)$ the connection-free part of $\mathbb{A}$.*

The curvature of a superconnection $\mathbb{A}$ is thus the sum of the curvature of the connection $\nabla^E := \mathbb{A}_{[1]}$, the exterior covariant derivative of its connection-free part $\bar{\mathbb{A}}$ with respect to ∇^E, and $\bar{\mathbb{A}}^2$.

Now we turn out attention to a specific class of superconnections on a Clifford module, namely the Clifford superconnections. Recall, that a Clifford module is a $\mathbb{Z}_2$-graded complex vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ over a Riemanniann manifold M together with a $\mathbb{Z}_2$-graded left action $c\colon C(M) \times \mathcal{E} \to \mathcal{E}$ of the Clifford bundle, i.e. a graded representation of $C(M)$ [1]). Generalizing the notion of a Clifford connection (cf. [5]), a Clifford superconnection on $\mathcal{E}$ is defined to be a superconnection $\mathbb{A}\colon \Gamma(\mathcal{E})^\pm \to \Omega^*(M, \mathcal{E})^\mp$ which is compatible with the Clifford action c, i.e. $[\mathbb{A}, c(a)] = c(\nabla a)$ for all $a \in \Gamma(C(M))$. In this formula, ∇ denotes the Levi-Civita connection extended to the Clifford bundle $C(M)$. Obviously any Clifford connection defines a Clifford superconnection with trivial connection-free part $\bar{\mathbb{A}} = 0$. Furthermore the connection part $\mathbb{A}_{[1]}$ of a Clifford superconnection $\mathbb{A}$ determines a Cifford connection $\nabla^{\mathcal{E}}$.

For later use we compare $\mathbf{c}(\bar{\mathbb{A}}^2)$ and $\mathbf{c}(\bar{\mathbb{A}})^2$ for the connection-free part $\bar{\mathbb{A}}$ of any Clifford superconnection $\mathbb{A}$. Here $\mathbf{c}\colon \Omega^*(M, \mathrm{End}\ \mathcal{E}) \to \Omega^0(M, \mathrm{End}\ \mathcal{E})$ denotes the obvious extension of the quantisation map $\mathbf{c}\colon \Lambda^*T^*M \to C(M)$. Note that this map $\mathbf{c}$ is not a homomorphism of algebras but yields the identity when restricted to $\Omega^0(M, \mathrm{End}\ \mathcal{E})$. Thus, by a simple calculation we get the following

LEMMA 2.2. *Let $\mathbb{A}$ be a Clifford superconnection on a Clifford module $\mathcal{E}$. Then $\mathbf{c}(\bar{\mathbb{A}})^2 - \mathbf{c}(\bar{\mathbb{A}}^2) = \sum_{i,j\geq 2} \big(\mathbf{c}(\mathbb{A}_{[i]})\mathbf{c}(\mathbb{A}_{[j]}) - \mathbf{c}(\mathbb{A}_{[i]}\mathbb{A}_{[j]})\big)$ where $\bar{\mathbb{A}} := \sum_{i\neq 1} \mathbb{A}_{[i]}$ denotes the 'connection-free' part of $\mathbb{A}$.*

PROOF: Because the quantisation map is linear we have $\mathbf{c}(\bar{\mathbb{A}}) = \sum_{i\neq 1} \mathbf{c}(\mathbb{A}_{[i]})$. Thus,

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[1]) For convenience of the reader and to fix our conventions we remark that $C(M)$ is the bundle of Clifford algebras over M generated by $T^*M_{\mathbb{C}} := T^*M \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the relations $v \star w + w \star v = -2g(v, w)$ for sections $v, w \in \Gamma(T^*M_{\mathbb{C}})$.

we compute

$$
\begin{aligned}
\mathbf{c}(\bar{\mathbb{A}})\mathbf{c}(\bar{\mathbb{A}}) &= \mathbf{c}(\mathbb{A}_{[0]}^2)+\\
&\quad [\mathbf{c}\big(\mathbb{A}_{[0]}\big),\mathbf{c}\big(\mathbb{A}_{[2]}\big)]+\\
&\quad [\mathbf{c}\big(\mathbb{A}_{[0]}\big),\mathbf{c}\big(\mathbb{A}_{[3]}\big)]+\\
&\quad [\mathbf{c}\big(\mathbb{A}_{[0]}\big),\mathbf{c}\big(\mathbb{A}_{[4]}\big)] + \mathbf{c}(\mathbb{A}_{[2]})^2+\\
&\quad \vdots\\
&\quad [\mathbf{c}\big(\mathbb{A}_{[0]}\big),\mathbf{c}\big(\mathbb{A}_{[i]}\big)] + [\mathbf{c}\big(\mathbb{A}_{[2]}\big),\mathbf{c}\big(\mathbb{A}_{[i-2]}\big)] + \ \ldots\\
&\quad \vdots
\end{aligned}
\tag{2.3}
$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the supercommutator in $\Omega^0(M,\mathrm{End}\,\mathcal{E})$. Because on the zero-level $\mathbf{c}(\mathbb{A}_{[0]}) = \mathbb{A}_{[0]}$ holds together with $\mathbb{A}_{[0]} \in \Gamma(\mathrm{End}_{C(M)}\ \mathcal{E})$ we get $\mathbf{c}\big(\mathbb{A}_{[0]}\mathbb{A}_{[i]}\big) = \mathbb{A}_{[0]}\mathbf{c}(\mathbb{A}_{[i]})$ for $i \neq 1$. In turn this implies $[\mathbf{c}\big(\mathbb{A}_{[0]}\big),\mathbf{c}\big(\mathbb{A}_{[i]}\big)] = \mathbf{c}\big([\mathbb{A}_{[0]},\mathbb{A}_{[i]}]\big)$ for all $i \neq 1$. Finally, by using $\mathbf{c}(\bar{\mathbb{A}}^2) = \sum_{i\neq 1}\ \mathbf{c}\big((\bar{\mathbb{A}}^2)_{[i]}\big)$ together with (2.2) we obtain the desired result.

$\square$

Thus, this difference $\big(\mathbf{c}(\bar{\mathbb{A}})^2 - \mathbf{c}(\bar{\mathbb{A}}^2)\big)$ is independent of the zero-degree part $\mathbb{A}_{[0]}$ of the Clifford superconnection $\mathbb{A}$. If n denotes the dimension of the underlying manifold M, obviously it is true that $(\bar{\mathbb{A}}^2)_{[i]} = 0$ for $i > n$ whereas in general $\mathbf{c}(\mathbb{A}_{[i]})\mathbf{c}(\mathbb{A}_{[j]}) \neq 0$ for $(i+j) > n$. Consequently in low dimensions it is $\sum_{i,j\geq 2}\ \mathbf{c}(\mathbb{A}_{[i]})\mathbf{c}(\mathbb{A}_{[j]})$ which mainly contributes to the above examined difference. For example, we have

$$
\begin{aligned}
\big(\mathbf{c}(\bar{\mathbb{A}})^2 - \mathbf{c}(\bar{\mathbb{A}}^2)\big) = \sum_{i=2}^{4}\mathbf{c}(\mathbb{A}_{[i]})^2 - \mathbf{c}(\mathbb{A}_{[2]}^2) + [\mathbf{c}\big(\mathbb{A}_{[2]}\big),\mathbf{c}\big(\mathbb{A}_{[3]}\big)]+\\
+ [\mathbf{c}\big(\mathbb{A}_{[2]}\big),\mathbf{c}\big(\mathbb{A}_{[4]}\big)] + [\mathbf{c}\big(\mathbb{A}_{[3]}\big),\mathbf{c}\big(\mathbb{A}_{[4]}\big)]
\end{aligned}
\tag{2.4}
$$

for a Clifford superconnection $\mathbb{A} := \sum_{i=0}^{4}\ \mathbb{A}_{[i]}$ on a Clifford module $\mathcal{E}$ over a four-dimensional Riemannian manifold M.

3. Some canonical constructions

Given a Clifford module $\mathcal{E}$ over a Riemannian mannifold M we establish in this section a canonical projection $\mathbf{g}\colon \Omega^*(M, \mathrm{End}\ \mathcal{E}) \to \Omega^1(M, \mathrm{End}\ \mathcal{E})$. This map enables us to relate a Clifford superconnection $\mathbb{A}\colon \Gamma(\mathcal{E}) \to \Omega^*(M, \mathcal{E})$ with an ordinary connection $\widetilde{\nabla}^{\mathcal{E}}$ which in general is not even a Clifford connection. However it turns out that this attribution $\mathbb{A} \mapsto \widetilde{\nabla}^{\mathcal{E}}$ preserves the most important information of $\mathbb{A}$ concerning our purpose [2]). For explaining this we first observe the

LEMMA 3.1. *Let* $\nabla^{\mathcal{E}}\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ *be a Clifford connection on the Clifford module* $\mathcal{E}$*. Then there exists a covariant constant section* $\sigma \in \Gamma(T^*M \otimes \mathrm{End}\mathcal{E})$ *such that* $c(\sigma) = 1\!\!1_{\mathcal{E}}$.

PROOF: Let $Sym^2(T^*M)$ denote the bundle of symmetric two tensors of T^*M over M. There are natural inclusions $Sym^2(T^*M) \overset{i}{\hookrightarrow} T^*M \otimes C(M) \overset{j}{\hookrightarrow} T^*M \otimes \mathrm{End}\mathcal{E}$ which are compatible with the induced connections, i.e.

$$j_*\big(i_*(\nabla_X s)\big) = j_*\big(\nabla_X^{T^*M\otimes C(M)} i_*(s)\big) = \nabla^{T^*M\otimes \mathrm{End}\mathcal{E}} j_*\big(i_*(s)\big) \tag{3.1}$$

for all sections $s \in \Gamma(Sym^2(T^*M))$ and all $X \in \Gamma(TM)$. Here the covariant derivative $\nabla\colon \Gamma(Sym^2(T^*M)) \to \Gamma(T^*M \otimes Sym^2(T^*M))$ is induced by the Levi-Civita connection on T^*M and the map $i_*\colon \Gamma(Sym^2(T^*M)) \to \Gamma(T^*M \otimes C(M))$ or $j_*\colon \Gamma(T^*M \otimes C(M)) \to \Gamma(T^*M \otimes \mathrm{End}\mathcal{E})$ denote push-forward, respectively. The second identity holds because we have $\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}} := \nabla \otimes 1\!\!1_{\mathcal{E}} + 1\!\!1_{T^*M} \otimes \nabla^{\mathrm{End}\mathcal{E}}$ with $\nabla^{\mathrm{End}\mathcal{E}}$ induced by the given Clifford connection $\nabla^{\mathcal{E}}$.

Now take a covariant constant section $\omega \in \Gamma(Sym^2(T^*M))$ which yields a non-zero constant r when evaluated with the Riemannian metric g, i.e $ev_g(\omega) = r$. Because of (3.1) it is true that $\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}} j_*(i_*\omega) = 0$ holds. Furthermore we get $(c \circ j_* \circ i_*) = c^2$ with the map $c^2\colon T^*M \otimes T^*M \to C(M)$ defined by $c^2(v \otimes w) := c(v)c(w)$ for all $v, w \in \Gamma(T^*M)$. Using the well-known identity $c^2(v \otimes w) = \mathbf{c}(v \wedge w) - ev_g(v \otimes w)$ where $\mathbf{c}\colon \Gamma(\Lambda^*T^*M) \to \Gamma(C(M))$ denotes the quantisation map this can further be simplified as $(c \circ j_* \circ i_*) = -ev_g$. So $\sigma := -\frac{1}{r}\, j_*(i_*\omega)$ has the desired properties.

□

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[2]) In fact we will show in the next section that the Dirac operators $D_{\mathbb{A}} := \mathbf{c}\circ\mathbb{A}$ and $\widetilde{D} := c\circ\widetilde{\nabla}^{\mathcal{E}}$ defined by the Clifford superconnection and the the corresponding connection, respectively, coincide.

With respect to a local coordinate system and using the Einstein convention we may also write $\sigma = -\frac{1}{r}\ \omega_{\mu\nu}\ dx^\mu \otimes c(dx^\nu)$ where the coefficients $\omega_{\mu\nu}$ are totally symmetric. Note that in general there may exist many sections of $T^*M \otimes \mathrm{End}\mathcal{E}$ with the above mentioned properties, so by no way we can achieve uniqueness of the End $\mathcal{E}$-vallued one-form σ in the previous lemma. However there is a canonical choice: If we take $g \in \Gamma(Sym^2(T^*M))$ the Riemann metric then, by the above construction, $\gamma := -\frac{1}{n}\ g_{\mu\nu}\ dx^\mu \otimes c(dx^\nu) \in \Omega^1(M, \mathrm{End}\mathcal{E})$ with $n := \dim\ M$ accomplishes $\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\gamma = 0$ and $c(\gamma) = 1\!\!1_{\mathcal{E}}$. In addition, γ is canonical with respect to the Riemannian structure of M.

For any element $\alpha \in \Omega^l(M, \mathrm{End}\mathcal{E})$ let $\mu(\alpha)\colon \Omega^k(M, \mathrm{End}\mathcal{E}) \to \Omega^{k+l}(M, \mathrm{End}\mathcal{E})$ denote multiplication in the graded algebra $\Omega^*(M, \mathrm{End}\mathcal{E})$. So the canonical one-form $\gamma \in \Omega^1(M, \mathrm{End}\mathcal{E})^+$ induces an even map $\mu(\gamma)\colon \Omega^*(M, \mathrm{End}\mathcal{E})^\pm \to \Omega^{*+1}(M, \mathrm{End}\mathcal{E})^\pm$. On the zero level $\mu(\gamma)\colon \Omega^0(M, \mathrm{End}\mathcal{E}) \to \Omega^1(M, \mathrm{End}\mathcal{E})$ is injective because Lemma 3.1 implies that $c \circ \mu(\gamma) = 1\!\!1_{\mathcal{E}}$ holds. We now define

$$\mathbf{g}\colon \Omega^*(M, \mathrm{End}\mathcal{E}) \xrightarrow{\mathbf{c}} \Omega^0(M, \mathrm{End}\mathcal{E}) \xrightarrow{\mu(\gamma)} \Omega^1(M, \mathrm{End}\mathcal{E}). \tag{3.2}$$

This map is linear and satisfies $c \circ \mathbf{g} = c \circ \mu(\gamma) \circ \mathbf{c} = 1\!\!1_{\mathcal{E}} \circ \mathbf{c} = \mathbf{c}$ by the above mentioned property of $\mu(\gamma)$. Consequently we have $\mathbf{g}^2 = \mathbf{g} \circ \mathbf{g} = \mu(\gamma) \circ c \circ \mathbf{g} = \mu(\gamma) \circ \mathbf{c} = \mathbf{g}$, so $\mathbf{g}$ is a projection. Let '$\cdot$' be the pointwise defined product in the algebra bundle $T(M) \otimes \mathrm{End}\ \mathcal{E}$, where $T(M)$ denotes the tensor bundle of T^*M and $i_X\colon \Omega^*(M, \mathrm{End}\ \mathcal{E}) \to \Omega^{*-1}(M, \mathrm{End}\ \mathcal{E})$ be the inner derivative with respect to $X \in \Gamma(TM)$. We will now study in greater detail this map $\mathbf{g}$. Moreover, the following two lemmas are essential in in proving our theorem.

Lemma 3.2. *Let $\mathcal{E}$ be a Clifford module over an even-dimensional Riemannian manifold M, $\nabla^{\mathcal{E}}$ be a Clifford connection, $d^{\nabla^{End\ \mathcal{E}}}\colon \Omega^*(M, \mathrm{End}\ \mathcal{E}) \to \Omega^{*+1}(M, \mathrm{End}\ \mathcal{E})$ be the induced exterior covariant derivative and $\alpha \in \Omega^*(M, \mathrm{End}\ \mathcal{E})^-$. Then the canonical map $\mathbf{g}\colon \Omega^*(M, \mathrm{End}\ \mathcal{E}) \to \Omega^1(M, \mathrm{End}\ \mathcal{E})$ has the properties*

$$\mathbf{c}\big(\mathbf{g}(\alpha)^2\big) + ev_g\big(\mathbf{g}(\alpha) \cdot \mathbf{g}(\alpha)\big) = \mathbf{c}(\alpha)^2 + 2ev_g\big(\beta(\alpha) \cdot \mathbf{g}(\alpha)\big) \tag{3.3}$$

$$c^2\big(\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\mathbf{g}(\alpha)\big) = \mathbf{c}\big(d^{\nabla^{End\mathcal{E}}}\alpha\big) - ev_g \nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\beta(\alpha) \tag{3.4}$$

where $\beta(\alpha) \in \Omega^1(M, \mathrm{End}\ \mathcal{E})$ is defined by $\beta(\alpha) := dx^k \otimes \mathbf{c}\big(i(\partial_k)\alpha\big)$ with respect to a local coordinate frame.

Proof: First identity (3.3): With respect to a local coordinate system we may write $\mathbf{g}(\alpha) = dx^\mu \otimes \mathbf{g}(\alpha)_\mu$. Thus, $\mathbf{c}\big(\mathbf{g}(\alpha)^2\big)$ equals $\frac{1}{4}\,[c(dx^\mu), c(dx^\nu)][\mathbf{g}(\alpha)_\mu, \mathbf{g}(\alpha)_\nu]$. Furthermore using $\frac{1}{4}\,[c(dx^\mu), c(dx^\nu)][\omega_\mu, \omega_\nu] = c(dx^\mu)\omega_\mu c(dx^\nu)\omega_\nu + g^{\mu\nu}\omega_\mu\omega_\nu - c(dx^\mu)[\omega_\nu, c(dx^\nu)]\omega_\nu$ which is true for any $\omega = dx^\mu \otimes \omega_\mu \in \Omega^1(M, \mathrm{End}\,\mathcal{E})$ we obtain

$$\begin{aligned} \mathbf{c}\big(\mathbf{g}(\alpha)^2\big) = c(dx^\mu)\mathbf{g}(\alpha)_\mu c(dx^\nu)\mathbf{g}(\alpha)_\nu + g^{\mu\nu}\mathbf{g}(\alpha)_\mu\mathbf{g}(\alpha)_\nu \\ - c(dx^\mu)[\mathbf{g}(\alpha)_\mu, c(dx^\nu)]\mathbf{g}(\alpha)_\nu. \end{aligned} \tag{3.5}$$

Because $c(dx^\mu)\mathbf{g}(\alpha)_\mu = (c\circ\mathbf{g})(\alpha) = \mathbf{c}(\alpha)$ it remains to show that the sum of the last two terms in (3.5) equals $2ev_g\big(\beta(\alpha)\cdot\mathbf{g}(\alpha)\big) - ev_g\big(\mathbf{g}(\alpha)\cdot\mathbf{g}(\alpha)\big)$. This is a consequence of the following lemma 3.3 .

Second identity (3.4): Using the definition of the map $\mathbf{g}$ and the compatibility condition $\nabla_X^{T^*M\otimes\mathrm{End}\mathcal{E}}\mu(\omega_1)\omega_0 = \mu\big(\nabla_X^{T^*M\otimes\mathrm{End}\mathcal{E}}\omega_1\big)\omega_0 + \mu(\omega_1)\big(\nabla_X^{\mathrm{End}\mathcal{E}}\omega_0\big)$ for all $\omega_i \in \Omega^i(M, \mathrm{End}\,\mathcal{E})$, $i = 0,1$ and $X \in \Gamma(TM)$ we compute

$$\begin{aligned} \nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\mathbf{g}(\alpha) &= dx^\mu \otimes \nabla_\mu^{T^*M\otimes\mathrm{End}\mathcal{E}}(\mu(\gamma)\mathbf{c}(\alpha)) \\ &= \underbrace{dx^\mu \otimes \mu\big(\nabla_\mu^{T^*M\otimes\mathrm{End}\mathcal{E}}\gamma\big)}_{\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\gamma=0}\mathbf{c}(\alpha) + dx^\mu \otimes \mu(\gamma)\nabla_\mu^{\mathrm{End}\mathcal{E}}\mathbf{c}(\alpha) \end{aligned} \tag{3.6}$$

Thus, we get $c^2\big(\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\mathbf{g}(\alpha)\big) = c(dx^\mu)c(\mu(\gamma))\nabla_\mu^{\mathrm{End}\mathcal{E}}\mathbf{c}(\alpha) = c(dx^\mu)\nabla_\mu^{\mathrm{End}\mathcal{E}}\mathbf{c}(\alpha)$ by the above mentioned property of $\mu(\gamma)$. Note, that because of being induced by a Clifford connection, $\nabla^{\mathrm{End}\mathcal{E}}$ is compatible with the quantisation map $\mathbf{c}$. More precisely it is true that $\nabla_X^{\mathrm{End}\mathcal{E}}\mathbf{c}(\omega) = \mathbf{c}\big(\nabla_X^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}}\omega\big)$ holds for all forms $\omega \in \Omega^*(M, \mathrm{End}\mathcal{E})$ and $X \in \Gamma(TM)$. Here the tensor product connection $\nabla^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}}$ is defined by $\nabla^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}} := \nabla^{\Lambda^*(T^*M)} \otimes 1\!\!1_{\mathcal{E}} + 1\!\!1_{\Lambda^*(T^*M)} \otimes \nabla^{\mathrm{End}\mathcal{E}}$ with the connetion $\nabla^{\Lambda^*(T^*M)}\colon \Gamma(\Lambda^*(T^*M)) \to \Gamma(T^*M \otimes \Lambda^*(T^*M))$ on the exterior bundle being induced by the Levi-Civita connection. This compatibility condition together with the equivariance of the quantisation map $\mathbf{c}\colon \Lambda^*(T^*M) \to C(M)$ with respect to the respective Clifford actions enables us to transform

$$\begin{aligned} c(dx^\mu)\nabla_\mu^{\mathrm{End}\mathcal{E}}\mathbf{c}(\alpha) &= c(dx^\mu)\mathbf{c}\big(\nabla_\mu^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}}\alpha\big) \\ &= \mathbf{c}\big(dx^\mu \wedge \nabla_\mu^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}}\alpha\big) - \mathbf{c}\big(g^{\mu\sigma}i(\partial_\sigma)\nabla_\mu^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}}\alpha\big) \end{aligned}$$

In the first term by definition $dx^\mu \wedge \nabla_\mu^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}} =: d^{\nabla^{End\mathcal{E}}}$ is the exterior covariant derivative. Thus, it only remains to look after the second one: Obviously the well-known identity $[\nabla_\mu, i(\partial_\sigma)] = i(\nabla_\mu\partial_\sigma)$ implies

$$[\nabla_\mu^{\Lambda^*(T^*M)\otimes\mathrm{End}\mathcal{E}}, i(\partial_\sigma)] = i(\nabla_\mu\partial_\sigma) = \Gamma^\nu_{\mu\sigma}i(\partial_\nu) \tag{3.7}$$

where $\Gamma^{\nu}_{\mu\sigma}$ denotes the Cristoffel symbols. Consequently we obtain

$$\begin{aligned}\mathbf{c}\big(g^{\mu\sigma} i(\partial_\sigma)\nabla_\mu^{\Lambda^*(T^*M)\otimes \mathrm{End}\mathcal{E}}\alpha\big) &= g^{\mu\sigma}\nabla_\mu^{\mathrm{End}\mathcal{E}}\mathbf{c}\big(i(\partial_\sigma)\alpha\big) - g^{\mu\sigma}\Gamma^{\nu}_{\mu\sigma}\mathbf{c}\big(i(\partial_\nu)\alpha\big)\\ &= g^{\mu\sigma}\nabla_\mu^{T^*M\otimes \mathrm{End}\mathcal{E}}\mathbf{c}\big(i(\partial_\sigma)\alpha\big)\end{aligned}$$

by using the compatibility of the connection $\nabla^{\mathrm{End}\mathcal{E}}$ with the quantisation map $\mathbf{c}$ in the inverse direction.

□

LEMMA 3.3. *Let $\alpha \in \Omega^*(M, \mathrm{End}\,\mathcal{E})^-$ and $\beta(\alpha) \in \Omega^1(M, \mathrm{End}\,\mathcal{E})^-$ be defined as in the previous lemma. Then $\beta(\alpha) = \mathbf{g}(\alpha) - \frac{1}{2}\, g_{\sigma\nu} dx^\sigma \otimes c(dx^\mu)[\mathbf{g}(\alpha)_\mu, c(dx^\nu)]$ holds with respect to a local coordinate frame.*

PROOF: Using the property $c \circ \mathbf{g} = \mathbf{c}$ of the canonical projection map $\mathbf{g}$ we obtain

$$\mathbf{g}(\alpha) - \frac{1}{2}\, g_{\sigma\nu} dx^\sigma \otimes c(dx^\mu)[\mathbf{g}(\alpha)_\mu, c(dx^\nu)] = dx^\sigma \otimes \Big(-\frac{1}{2}\, g_{\sigma\nu}[c(dx^\nu), \mathbf{c}(\alpha)]\Big). \quad (3.8)$$

Note that $[\,\cdot\,,\,\cdot\,]$ denotes the supercomutator in $\Omega^0(M, \mathrm{End}\,\mathcal{E})$. Thus, it remains to show $-\frac{1}{2}\, g_{\sigma\nu}[c(dx^\nu), \mathbf{c}(\alpha)] = \mathbf{c}\big(i(\partial_\sigma)\alpha\big)$. Because $\mathrm{End}\,\mathcal{E} \cong C(M)\hat{\otimes}\mathrm{End}_{C(M)}\mathcal{E}$ this identity in $\mathrm{End}\,\mathcal{E}$ ensues from the following diagramm

$$\begin{array}{ccc}\Lambda^*(T^*M) & \xrightarrow{i(X)} & \Lambda^*(T^*M)\\ \big\downarrow \mathbf{c} & & \big\downarrow \mathbf{c}\\ C(M) & \xrightarrow{-\frac{1}{2}\,[c(X^*),\,\cdot\,]} & C(M)\end{array} \quad (3.9)$$

which is commutative for all $X \in \Gamma(TM)$. Here $X^* := g(X, \,\cdot\,) \in \Gamma(T^*M)$ denotes the dual of the vectorfield X and $[\,\cdot\,,\,\cdot\,]$ is the supercommutator in the Clifford bundle $C(M)$.

□

Thus, $-c(dx^\mu)[\mathbf{g}(\alpha)_\mu, c(dx^\nu)]\mathbf{g}(\alpha)_\nu = 2g^{\sigma\nu}\mathbf{c}\big(i(\partial_\sigma)\alpha\big)\mathbf{g}(\alpha)_\nu - 2g^{\sigma\nu}\mathbf{g}(\alpha)_\sigma\mathbf{g}(\alpha)_\nu$ holds and completes the proof of equation (3.3) in Lemma 3.2 .

In the following we denote by $\mathcal{C}(\mathcal{E})$ the collection of all connections and by $\mathcal{CSC}(\mathcal{E})$ the collection of all Clifford superconnections on a Clifford module $\mathcal{E}$. It is well-known that $\mathcal{C}(\mathcal{E})$ and $\mathcal{CSC}(\mathcal{E})$ are affine spaces modelled over $\Omega^1(M, \mathrm{End}\,\mathcal{E})$ and

$\Omega^*(M, \mathrm{End}_{C(M)}\mathcal{E})$, respectively. With the map $\mathbf{g}\colon \Omega^*(M, \mathrm{End}\ \mathcal{E}) \to \Omega^1(M, \mathrm{End}\ \mathcal{E})$ in hand we are able to define an affine map

$$
\begin{aligned}
\mathcal{CSC}(\mathcal{E}) &\longrightarrow \mathcal{C}(\mathcal{E}) \\
\mathbb{A} &\longmapsto \nabla^{\mathbb{A}} := \nabla^{\mathcal{E}} + \mathbf{g}(\bar{\mathbb{A}}).
\end{aligned}
\tag{3.10}
$$

As before (cf. section 2) we use the notation $\nabla^{\mathcal{E}} := \mathbb{A}_{[1]}$ for the connection part and $\bar{\mathbb{A}}$ for the connection-free part of $\mathbb{A}$. Now we compare the corresponding curvature $R^{\nabla^{\mathbb{A}}}$ with $\mathbb{F}(\mathbb{A})$: By definition, resp. lemma 2.1 we have

$$
R^{\nabla^{\mathbb{A}}} = R^{\nabla^{\mathcal{E}}} + d^{\nabla^{End\ E}}\mathbf{g}(\bar{\mathbb{A}}) + \mathbf{g}(\bar{\mathbb{A}})^2 \tag{3.11}
$$

$$
\mathbb{F}(\mathbb{A}) = R^{\nabla^{\mathcal{E}}} + d^{\nabla^{End\ E}}\bar{\mathbb{A}} + (\bar{\mathbb{A}})^2, \tag{3.12}
$$

and therefore $\big(R^{\nabla^{\mathbb{A}}} - \mathbb{F}(\mathbb{A})\big) = (d^{\nabla^{End\ E}}\mathbf{g}(\bar{\mathbb{A}}) - d^{\nabla^{End\ E}}\bar{\mathbb{A}}) + (\mathbf{g}(\bar{\mathbb{A}})^2 - \bar{\mathbb{A}}^2)$. Thus, using lemma 3.2 we obtain the

Corollary 3.4. *Let $\mathbb{A}$ be a Clifford superconnection on a Clifford module $\mathcal{E}$ and $\nabla^{\mathbb{A}}$ the associated connection with respect to the affine map (3.10). Then*

$$
\begin{aligned}
\mathbf{c}\big(R^{\nabla^{\mathbb{A}}} - \mathbb{F}(\mathbb{A})\big) = \big(\mathbf{c}(\bar{\mathbb{A}})^2 - \mathbf{c}(\bar{\mathbb{A}}^2)\big) + ev_g\Big(\nabla^{T^*M\otimes\mathrm{End}\mathcal{E}}\big(\mathbf{g}(\bar{\mathbb{A}}) - \beta(\bar{\mathbb{A}})\big)\Big) + \\
+ 2ev_g\big(\beta(\bar{\mathbb{A}})\cdot\mathbf{g}(\bar{\mathbb{A}})\big) - ev_g\big(\mathbf{g}(\bar{\mathbb{A}})\cdot\mathbf{g}(\bar{\mathbb{A}})\big)
\end{aligned}
$$

holds where $\mathbb{F}(\mathbb{A}) := \mathbb{A}^2$ denotes the supercurvature and $R^{\nabla^{\mathbb{A}}} := (\nabla^{\mathbb{A}})^2$ the curvature of $\mathbb{A}$ and $\nabla^{\mathbb{A}}$, respectively.

4. The generalized Lichnerowicz formula

Let $\mathcal{E}$ be a Clifford module over an even-dimensonal Riemannian manifold M. Generalizing Dirac's original notion a Dirac operator acting on sections of $\mathcal{E}$ can be defined as an odd-parity first order differential operator $D\colon \Gamma(\mathcal{E}^{\pm}) \to \Gamma(\mathcal{E}^{\mp})$ such that its square D^2 is a generalized laplacian (cf. [5]). We will regard only those Dirac operators D that are compatible with the given Clifford module structure on $\mathcal{E}$, i.e. $[D, f] = c(df)$ holds for all $f \in C^\infty(M)$. Note that this property fully characterizes

those Dirac operators. Given any superconnection $\mathbb{A}\colon \Gamma(\mathcal{E})^{\pm} \to \Omega^*(M,\mathcal{E})^{\mp}$ on $\mathcal{E}$, the first order operator $D_{\mathbb{A}}$ defined by the following composition

$$\Gamma(\mathcal{E}^{\pm}) \xrightarrow{\mathbb{A}} \Omega^*(M,\mathcal{E})^{\mp} \xrightarrow{\cong} \Gamma(C(M)\otimes\mathcal{E})^{\mp} \xrightarrow{c} \Gamma(\mathcal{E}^{\mp}) \tag{4.1}$$

obviously is a Dirac operator. Here the isomorphism is induced by the quantisation map $\mathbf{c}\colon \Lambda^*(T^*M)\xrightarrow{\cong} C(M)$ and the last map denotes the given Clifford action of $C(M)$ on $\mathcal{E}$. Note that any connection $\widetilde{\nabla}^{\mathcal{E}}\colon \Gamma(\mathcal{E}^{\pm}) \to \Gamma(T^*M\otimes\mathcal{E}^{\pm})$ which respects the grading is also a superconnection. Moreover, as it is shown in [5], due to the above construction (4.1) any Dirac operator is uniquely determined by a Clifford superconnection, i.e. the assignment $\mathbb{A}\mapsto D_{\mathbb{A}}$ for $\mathbb{A}\in\mathcal{CSC}(\mathcal{E})$ is a bijection.

Going back to the map (3.10) we can associate a connection $\nabla^{\mathbb{A}}\in\mathcal{C}(\mathcal{E})$ to any Clifford superconnection $\mathbb{A}\in\mathcal{CSC}(\mathcal{E})$ and we are interessted in comparing the corresponding Dirac operators:

Lemma 4.1. *Let $\mathbb{A}$ be a Clifford superconnection on a Clifford module $\mathcal{E}$ and $\nabla^{\mathbb{A}}$ the associated connection with respect to the affine map (3.10). Then the corresponding Dirac operators $D_{\mathbb{A}}$ and $D_{\nabla^{\mathbb{A}}}$ coincide.*

Proof: Because we have defined $\nabla^{\mathbb{A}} := \nabla^{\mathcal{E}} + \mathbf{g}(\bar{\mathbb{A}})$ where $\nabla^{\mathcal{E}} := \mathbb{A}_{[1]}$ denotes the connection part of the Clifford superconnection $\mathbb{A}$ we obviously obtain

$$D_{\nabla^{\mathbb{A}}} := c\circ\nabla^{\mathbb{A}} = c\circ\mathbb{A}_{[1]} + (c\circ\mathbf{g})(\bar{\mathbb{A}}) = c\circ\mathbb{A}_{[1]} + \mathbf{c}(\bar{\mathbb{A}}) = \mathbf{c}(\mathbb{A}) =: D_{\mathbb{A}}.$$

Here we have used the property $c\circ\mathbf{g} = \mathbf{c}$ of the canonical projection $\mathbf{g}$, cf. the previous section 3.

□

Thus, for any Dirac operator $D_{\mathbb{A}}$ on a Clifford module $\mathcal{E}$ there exists a connection $\widetilde{\nabla}^{\mathcal{E}}\colon \Gamma(\mathcal{E}^{\pm}) \to \Gamma(T^*M\otimes\mathcal{E}^{\pm})$ such that $D_{\widetilde{\nabla}^{\mathcal{E}}} = D_{\mathbb{A}}$. This is a restatement of Quillen's principle that 'Dirac operators are a quantisation of the theory of connections' (cf. the introduction of [5]). Note that for the same reason as we remark after lemma 3.1 we can not achieve uniqueness of the connection $\widetilde{\nabla}^{\mathcal{E}}$. However, the above defined connection $\nabla^{\mathbb{A}}$ is the canonical choice.

If $\widetilde{\nabla}^{\mathcal{E}}\colon \Gamma(\mathcal{E}^{\pm}) \to \Gamma(T^*M\otimes\mathcal{E}^{\pm})$ is a connection on a Clifford module $\mathcal{E}$, recently it has been shown that there is the following decomposition formula for the square of the corresponding Dirac operator $D_{\widetilde{\nabla}^{\mathcal{E}}} := c\circ\widetilde{\nabla}^{\mathcal{E}}$ (cf. [2]):

$$D^2_{\widetilde{\nabla}^{\mathcal{E}}} = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \mathbf{c}(R^{\widetilde{\nabla}^{\mathcal{E}}}) + ev_g\widetilde{\nabla}^{T^*M\otimes\mathrm{End}\mathcal{E}}\varpi_{\widetilde{\nabla}^{\mathcal{E}}} + ev_g(\varpi_{\widetilde{\nabla}^{\mathcal{E}}}\cdot\varpi_{\widetilde{\nabla}^{\mathcal{E}}}). \tag{4.2}$$

Here $\varpi_{\widetilde{\nabla}^{\mathcal{E}}} := -\frac{1}{2} g_{\nu\kappa} dx^\nu \otimes c(dx^\mu)\big([\widetilde{\nabla}^{\mathcal{E}}_\mu, c(dx^\kappa)] + c(dx^\sigma)\Gamma^\kappa_{\sigma\mu}\big) \in \Omega^1(M, \mathrm{End}\mathcal{E})$ indicates the deviation of the connection $\widetilde{\nabla}^{\mathcal{E}}$ being a Clifford connection and $\triangle^{\hat{\nabla}^{\mathcal{E}}}$ is the connection laplacian associated to $\hat{\nabla}^{\mathcal{E}} := \widetilde{\nabla}^{\mathcal{E}} + \varpi_{\widetilde{\nabla}^{\mathcal{E}}}$ [3]). Only the second term which denotes the image of the curvature $R^{\widetilde{\nabla}^{\mathcal{E}}} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ of the given connection $\widetilde{\nabla}^{\mathcal{E}}$ under the quantisation map $\mathbf{c}\colon \Lambda^* T^*M \to C(M)$, is endowed with geometric significance. Of course, if $\widetilde{\nabla}^{\mathcal{E}}$ is a Clifford connection, obviously $\varpi_{\widetilde{\nabla}^{\mathcal{E}}} = 0$ and therefore (4.2) reduces to Lichnerowicz's formula $D^2_{\widetilde{\nabla}^{\mathcal{E}}} = \triangle^{\widetilde{\nabla}^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}(R^{\mathcal{E}/S}_{\widetilde{\nabla}^{\mathcal{E}}})$. Because any Clifford superconnection $\mathbb{A} \in \mathcal{CSC}(\mathcal{E})$ uniquely determines a Dirac operator $D_{\mathbb{A}}$ as already mentioned above, it is natural to reformulate the generalized Lichnerowicz formula (4.2):

Theorem 4.2. *Let $\mathbb{A} = \mathbb{A}_{[1]} + \bar{\mathbb{A}}$ be a Clifford superconnection on a Clifford module $\mathcal{E}$ over an even-dimensional Riemannian manifold M and let $D_{\mathbb{A}} := \mathbf{c} \circ \mathbb{A}$ denote the corresponding Dirac operator. Then*

$$D^2_{\mathbb{A}} = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big) + \mathbf{c}\big(\bar{\mathbb{A}}\big)^2 - \mathbf{c}\big(\bar{\mathbb{A}}^2\big) + ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big) \tag{4.3}$$

where $\hat{\nabla}^{\mathcal{E}} := \mathbb{A}_{[1]} + \beta(\bar{\mathbb{A}})$ determines the connection laplacian $\triangle^{\hat{\nabla}^{\mathcal{E}}}$, $\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}$ denotes the twisting supercurvature of $\mathbb{A} \in \mathcal{CSC}(\mathcal{E})$ and $\beta(\bar{\mathbb{A}}) \in \Omega^1(M, \mathrm{End}\,\mathcal{E})$ is defined by $\beta(\bar{\mathbb{A}}) := dx^k \otimes \mathbf{c}\big(i(\partial_k)\bar{\mathbb{A}}\big)$ with respect to a local coordinate frame.

Proof: For convinience, let $\nabla^{\mathcal{E}} := \mathbb{A}_{[1]}$ denote the Clifford connection part of the Clifford superconnection $\mathbb{A}$. Lemma 4.1 tells us that the Dirac operator $D_{\mathbb{A}}$ corresponding to $\mathbb{A}$ can be equivalently obtained by $D_{\mathbb{A}} = c \circ \nabla^{\mathbb{A}}$ using the associated connection $\nabla^{\mathbb{A}} := \nabla^{\mathcal{E}} + \mathbf{g}(\bar{\mathbb{A}})$. Thus, we reformulate the decomposition formula (4.2) (cf. [2]):

$$D^2_{\mathbb{A}} = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \mathbf{c}(R^{\nabla^A}) + ev_g \bar{\nabla}^{T^*M \otimes \mathrm{End}\mathcal{E}} \varpi_{\nabla^A} + ev_g(\varpi_{\nabla^A} \cdot \varpi_{\nabla^A}). \tag{4.4}$$

Here we have $\bar{\nabla}^{T^*M \otimes \mathrm{End}\,\mathcal{E}} := \nabla \otimes \mathbb{1}_{\mathcal{E}} + \mathbb{1}_{T^*M} \otimes \widetilde{\nabla}^{\mathbb{A}}$ where ∇ denotes the Levi-Civita connection on T^*M and the connection $\widetilde{\nabla}^{\mathbb{A}}\colon \Gamma(\mathrm{End}\mathcal{E}) \to \Gamma(T^*M \otimes \mathrm{End}\mathcal{E})$ on the endomorphism bundle is induced by $\nabla^{\mathbb{A}}$.

Now we inspect the right hand side of formula (4.4) term by term: Recall that $\nabla^{\mathcal{E}}$ is compatible with the Clifford action, i.e. $[\nabla^{\mathcal{E}}_\mu, c(dx^\kappa)] = -c(dx^\sigma)\Gamma^\kappa_{\sigma\mu}$ holds with

---

[3]) With respect to a local coordinate frame of TM, the connection laplacian $\triangle^{\hat{\nabla}^{\mathcal{E}}}$ is explicitly given by $\triangle^{\hat{\nabla}^{\mathcal{E}}} = -g^{\mu\nu}(\hat{\nabla}^{\mathcal{E}}_\mu \hat{\nabla}^{\mathcal{E}}_\nu - \Gamma^\sigma_{\mu\nu}\hat{\nabla}^{\mathcal{E}}_\sigma)$.

respect to a local coordinate frame. This implies

$$
\begin{aligned}
\varpi_{\nabla^A} &:= -\frac{1}{2}\, g_{\nu\kappa} dx^\nu \otimes c(dx^\mu)[(\nabla^{\mathbb{A}}_\mu - \nabla^{\mathcal{E}}_\mu), c(dx^\kappa)] \\
&= -\frac{1}{2}\, g_{\nu\kappa} dx^\nu \otimes c(dx^\mu)[\mathbf{g}(\bar{\mathbb{A}})_\mu), c(dx^\kappa)]
\end{aligned}
\tag{4.5}
$$

and therefore $\hat{\nabla}^{\mathcal{E}} := \nabla^{\mathbb{A}} + \varpi_{\nabla^A} = \nabla^{\mathcal{E}} + \big(\mathbf{g}(\bar{\mathbb{A}}) - \frac{1}{2}\, g_{\nu\kappa} dx^\nu \otimes c(dx^\mu)[\mathbf{g}(\bar{\mathbb{A}})_\mu), c(dx^\kappa)]\big)$ holds. Thus, using lemma 3.3 it is true that $\hat{\nabla}^{\mathcal{E}} = \mathbb{A}_{[1]} + \beta(\bar{\mathbb{A}})$ holds with $\beta(\bar{\mathbb{A}}) \in \Omega^1(M, \mathrm{End}\,\mathcal{E})$ locally defined by $\beta(\bar{\mathbb{A}}) := dx^k \otimes \mathbf{c}\big(i(\partial_k)\bar{\mathbb{A}}\big)$.

Before we replace the curvature term $\mathbf{c}\big(R^{\nabla^A}\big)$ by $\mathbf{c}\big(\mathbb{F}(\mathbb{A})\big) + \mathbf{c}\Big(R^{\nabla^A} - \mathbb{F}(\mathbb{A})\Big)$ which involves the supercurvature, we study the last two terms in (4.4). Note that, again by lemma 3.3, we have $\varpi_{\nabla^A} = \beta(\bar{\mathbb{A}}) - \mathbf{g}(\bar{\mathbb{A}})$. Applying $\bar{\nabla}^{T^*M\otimes\mathrm{End}\,\mathcal{E}}_\mu = \nabla^{T^*M\otimes\mathrm{End}\,\mathcal{E}}_\mu + [\mathbf{g}(\bar{\mathbb{A}})_\mu,\ \cdot\ ]$ where $[\ \cdot\ ,\ \cdot\ ]$ denotes the commutator in End $\mathcal{E}$ and the tensor connection $\nabla^{T^*M\otimes\mathrm{End}\,\mathcal{E}} := \nabla \otimes \mathbb{1}_{\mathcal{E}} + \mathbb{1}_{T^*M} \otimes \nabla^{\mathcal{E}}$ is defined as in the proof of lemma 3.1 we obtain for the third term

$$
\begin{aligned}
ev_g \bar{\nabla}^{T^*M\otimes\mathrm{End}\mathcal{E}} \varpi_{\nabla^A} = ev_g &\Big(\nabla^{T^*M\otimes\mathrm{End}\,\mathcal{E}}\big(\beta(\bar{\mathbb{A}}) - \mathbf{g}(\bar{\mathbb{A}})\big)\Big) + \\
&+ ev_g\big(\mathbf{g}(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big) - ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \mathbf{g}(\bar{\mathbb{A}})\big).
\end{aligned}
\tag{4.6}
$$

For the forth one we calculate

$$
\begin{aligned}
ev_g(\varpi_{\nabla^A} \cdot \varpi_{\nabla^A}) &= ev_g\Big(\big(\beta(\bar{\mathbb{A}}) - \mathbf{g}(\bar{\mathbb{A}})\big) \cdot \big(\beta(\bar{\mathbb{A}}) - \mathbf{g}(\bar{\mathbb{A}})\big)\Big) \\
&= ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big) - ev_g\big(\mathbf{g}(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big) - \\
&\quad - ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \mathbf{g}(\bar{\mathbb{A}})\big) + ev_g\big(\mathbf{g}(\bar{\mathbb{A}}) \cdot \mathbf{g}(\bar{\mathbb{A}})\big),
\end{aligned}
\tag{4.7}
$$

always provided that the dot '$\cdot$' indicates the fibrewise defined product in the algebra bundle $T(M) \otimes \mathrm{End}\,\mathcal{E}$ with $T(M)$ being the tensor bundle of T^*M. Clearly,

$$
\begin{aligned}
(4.6) + (4.7) = ev_g&\Big(\nabla^{T^*M\otimes\mathrm{End}\,\mathcal{E}}\big(\beta(\bar{\mathbb{A}}) - \mathbf{g}(\bar{\mathbb{A}})\big)\Big) - 2ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \mathbf{g}(\bar{\mathbb{A}})\big) + \\
&+ ev_g\big(\mathbf{g}(\bar{\mathbb{A}}) \cdot \mathbf{g}(\bar{\mathbb{A}})\big) + ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big)
\end{aligned}
\tag{4.8}
$$

If we additionally insert $\mathbf{c}\big(\mathbb{F}(\mathbb{A})\big) + \mathbf{c}\Big(R^{\nabla^A} - \mathbb{F}(\mathbb{A})\Big)$ for $\mathbf{c}\big(R^{\nabla^A}\big)$ in formula (4.4) and use corollary 3.4, the first three terms of (4.8) cancel out. Finally, by using proposition 3.43 of [BGV] the supercurvature $\mathbb{F}(\mathbb{A})$ decomposes under the isomorphism End $\mathcal{E} \cong C(M) \otimes$ End $\mathcal{E}$ as $\mathbb{F}(\mathbb{A}) = c(R) + \mathbb{F}(\mathbb{A})^{\mathcal{E}/S}$ where $c(R) \in$

$\Omega^2(M, C(M))$ is the action of the Riemannian curvature R of M on the Clifford module and $\mathbb{F}(\mathbb{A})^{\mathcal{E}/S} \in \Omega^*(M, \mathrm{End}_{C(M)}\mathcal{E})$ denotes the twisting supercurvature of $\mathbb{A}$. This completes the proof of the theorem.

□

Obviously, the endomorphism $P(\bar{\mathbb{A}}) \in \Gamma(\mathrm{End}\ \mathcal{E})$ defined by the last three terms in the decomposition formula $P(\bar{\mathbb{A}}) := \mathbf{c}\big(\bar{\mathbb{A}}\big)^2 - \mathbf{c}\big(\bar{\mathbb{A}}^2\big) + ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big)$ depends only on the higher degree parts $\mathbb{A}_{[i]},\ i \geq 2$ of the Clifford superconnection $\mathbb{A}$. This means $P(\bar{\mathbb{A}}) = 0$ for $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]}$. Hence the generalized Lichnerowicz formula reduces to $D^2_{\mathbb{A}} = \triangle^{\nabla^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big)$ in this case. Furthermore, if we denote $\mathbb{A}_{[0]} := \Phi \in \mathrm{End}^-_{C(M)}\mathcal{E}$, we obtain

$$D^2_{(\Phi+\nabla^{\mathcal{E}})} = \triangle^{\nabla^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}(R^{\mathcal{E}/S}_{\nabla^{\mathcal{E}}}) + c\ \nabla^{\mathrm{End}\mathcal{E}}(\Phi) + \Phi^2. \tag{4.9}$$

Thus, we recover the decomposition formula for the square of a 'Dirac operator of simple type', cf. [2].

As we have already mentioned in the introduction, Getzler has stated the generalized Lichnerowicz formula in the form $D^2_{\mathbb{A}} = \triangle^{\hat{\nabla}^{\mathcal{E}}} + \frac{r_M}{4} + \mathbf{c}\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big) + P(\mathbb{A})$. In contrast to our result (4.3) above, he has not specified the endomorphism $P(\mathbb{A}) \in \Gamma(\mathrm{End}\ \mathcal{E})$ in general. However, calculation of $P(\mathbb{A})$ for $\mathbb{A} := \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$ where the two-form part is given by $\mathbb{A}_{[2]} := \frac{1}{2}\ dx^i \wedge dx^j \otimes \omega_{ij}$ with $\omega_{ij} \in \mathrm{End}_{C(M)}\mathcal{E}$ for all $i,\ j$ yields

$$P(\mathbb{A}) = 2g^{ij}\ \mathbf{c}(dx^k \wedge dx^l)\ \omega_{ik}\omega_{jl} - g^{ij}g^{kl}\ \omega_{ik}\omega_{jl} \tag{4.10}$$

which reproduces Getzler's example (cf. [7]).

5. Families of Dirac operators and the extended generalized Lichnerowicz formula

Suppose that $\pi\colon M \to B$ is a family of oriented even-dimensional Riemannian manifolds $(M_z \mid z \in B)$ and $\mathcal{E}$ is a bundle over M such that its fibrewise restriction $\mathcal{E}_z := \mathcal{E}|_z$ is a Clifford module for each $z \in B$. Furthermore, let $\nabla^{\mathcal{E}}$ be a connection on $\mathcal{E}$ which is a Clifford connection when restricted to each Clifford module $\mathcal{E}_z$ and let $\mathsf{D} := (\mathsf{D}^z \mid z \in B)$ be the associated family of Dirac operators. In [4], Bismut

constructed a superconnection ∇ on the C^∞-direct image $\pi_*\mathcal{E}$ corresponding to the family D whose Chern character, by using Getzler's rescaling trick, is explicitly computable. A crucial step in this calculation was the remarkably simple formula for the supercurvature ∇^2 (cf. [4] and [5], chapter 10.3) which can be understood as extending the ordinary Lichnerowicz formula to this infinit-dimensional context. In this section we follow Bismut's construction with $\nabla^{\mathcal{E}}$ replaced by an arbitrary superconnection $\mathbb{A}$ on $\mathcal{E}$ whose restriction to each bundle $\mathcal{E}_z$ is a Clifford superconnection. This enables us to introduce a superconnection $\nabla^{\mathbb{A}}$ on $\pi_*\mathcal{E}$ corresponding to the family of Dirac operators $\mathsf{D}_{\mathbb{A}} := (\mathsf{D}^z_{\mathbb{A}} \mid z \in B)$ defined by $\mathbb{A}$. For the supercurvature $(\nabla^{\mathbb{A}})^2$ we also obtain a simple formula. Similarly, that extends the above generalisation (4.3) of Lichnerowicz's decomposition.

First, we recall briefly the geometric structure (cf. [5]): Let $\pi\colon M \to B$ as above with a metric $g_{M/B}$ on the vertical tangent bundle $T(M/B)$ and a connection on TM. This induces a decomposition $TM = T(M/B) \oplus \pi^*TB$ where we have identified π^*TB with the horizontal space. Let $\nabla^{M/B}$ denote the connection on $T(M/B)$ associated to the vertical metric $g_{M/B}$ as constructed in [4]. Given a metric g_B on the base B with associated Levi-Civita connection ∇^B on TB one is able to define $\nabla^\oplus := \nabla^{M/B} \oplus \pi^*\nabla^B$. Note that $\nabla^\oplus$ preserves the metric $g = \pi^*g_B \oplus g_{M/B}$ but differs from the corresponding Levi-Civita connection ∇^g by a torsion-term, i.e. $(\nabla^g - \nabla^\oplus) \in \Omega^1(M, Sk(TM))$ where $Sk(TM)$ denotes the bundle of skew-symmetric endomorphisms of TM. Using the isomorphism $\tau\colon \Lambda^2TM \xrightarrow{\cong} Sk(TM)$ which is given by the Riemannian metric g we have $\nabla^g = \nabla^\oplus + \frac{1}{2}\,\tau(\omega)$. Interessting, this form $\omega \in \Omega^1(M, \Lambda^2TM)$ is independent of the choosen metric g_B, cf. [4]. Thus, using the blow-up metric $g_u := u^{-1}\pi^*g_B + g_{M/B}$ with $u \in \mathbb{R} \setminus \{0\}$, the limit as $u \to 0$ of the corresponding Levi-Civita connections $\nabla^{M,u}$ exits and yields

$$\nabla^{M,0} \;:=\; \lim_{u\to 0}\; \nabla^{M,u} \;=\; \nabla^g, \tag{5.1}$$

cf. [B] or [BGV]. Now let $g^u = u\pi^*g_B \oplus g_{M/B}$ denote the corresponding dual metric and $\nabla^{T^*M,u}, \nabla^{T^*M,0}$ the corresponding dual connections on T^*M. Note that the limit $g^0 := \lim_{u\to 0}\; g^u$ is degenerate on the horizontal space π^*T^*B. Furthermore, suppose that $\mathcal{E}$ is a hermitian vector bundle over M which, in addition, is a Clifford module along the fibres of the bundle (M, B, π). More precisely, there is a Clifford action $c\colon C(M/B) \to \operatorname{End}\mathcal{E}$ where $C(M/B)$ denotes the bundle of Clifford algebras over M generated by the vertical bundle $(T^*(M/B), g_{M/B})$. We now define a natural

action m_0 of $C_0(M) := \pi^*\Lambda T^*B \otimes C(M/B)$ on the bundle $\mathbb{E} := \pi^*\Lambda T^*B \otimes \mathcal{E}$ over M:

$$m_0(a) := \begin{cases} \epsilon(a) \otimes id_{\mathcal{E}} & \text{iff } a \in \pi^*T^*B \\ id_{\pi^*\Lambda T^*B} \otimes c(a) & \text{iff } a \in T^*(M/B). \end{cases} \tag{5.2}$$

Here ϵ denotes exterior multiplication on $\pi^*\Lambda T^*B$. Note that this action m_0 can be understood as the limit $\lim_{u\to 0}\ m_u$ of the Clifford actions m_u of the Clifford bundles $C_u(M)$ generated by T^*M with respect to the relation $v \star w + w \star v = -2g^u(v,w)$ for all $v, w \in \Gamma(T^*M)$ on $\mathbb{E}$ as defined in [5]. Thus, in this reference m_0 is called 'degenerate Clifford action'.

Now let $\mathbb{A}\colon \Gamma(\mathcal{E}^{\pm}) \to \Omega^*(M,\mathcal{E})^{\mp}$ be a superconnection which is a Clifford superconnection with respect to the above defined Clifford action c of the vertical Clifford bundle $C(M/B)$, i.e. $[\mathbb{A}, c(a)] = c(\nabla^{M/B}a)$ holds for all $a \in C(M/B)$. Then

$$\mathbb{A}^{\mathbb{E},\oplus} := \pi^*\nabla^B \otimes 1\!\!1_{\mathcal{E}} + 1\!\!1_{\pi^*\Lambda T^*B} \otimes \mathbb{A} \tag{5.3}$$

$$\mathbb{A}^{\mathbb{E},u} := \mathbb{A}^{\mathbb{E},\oplus} + \frac{1}{2}\, m_u(\omega) \tag{5.4}$$

are superconnections on $\mathbb{E}$ where $\omega \in \Omega^1(M, \Lambda^2T^*M)$ denotes the above mentioned torsion term considered as operating on $\mathbb{E}$ by the Clifford action m_u [4]. Here $\mathbf{m}_u\colon \Lambda T^*M \to C_u(M)$ denotes the respective quantisation maps. Obviously, by a similar argument as in Proposition 10.10 of [BGV], the various superconnections $\mathbb{A}^{\mathbb{E},u}$ are Clifford superconnections on $\mathbb{E}$ with respect to the corresponding Clifford actions m_u, i.e. $[\mathbb{A}^{\mathbb{E},u}, m_u(a)] = m_u(\nabla^{T^*M,u}a)$ for all $a \in C_u(M)$ holds. Let $D_{\mathbb{A},u} := \mathbf{m}_\mathrm{u} \circ \mathbb{A}^{\mathbb{E},u}$ denote the corresponding Dirac operators. Furthermore, taking the limit

$$\mathbb{A}^{\mathbb{E},0} \ := \ \lim_{u\to 0}\ \mathbb{A}^{\mathbb{E},u} \ = \ \mathbb{A}^{\mathbb{E},\oplus} + \frac{1}{2}\, m_0(\omega) \tag{5.5}$$

we obtain a superconnection with the property $[\mathbb{A}^{\mathbb{E},0}, m_0(a)] = m_0(\nabla^{T^*M,0}a)$ for all $a \in C_0(M)$. In other words, $\mathbb{A}^{\mathbb{E},0}$ respects the degenerate Clifford action m_0.

Recall that given a vector bundle $\mathcal{E}$ over M, the C^∞-direct image $\pi_*\mathcal{E}$ is the infinite-dimensional bundle over B whose fibre at $z \in B$ is defined to be the space $\Gamma(\mathcal{E}_z)$ of all C^∞-sections of the bundle $\mathcal{E}_z$ over M_z. Imitating the key idea in Bismut's construction we use the isomorphism

$$\Omega^*(B, \pi_*\mathcal{E}) \ \cong \ \Gamma(\pi^*\Lambda T^*B \otimes \mathcal{E}) \tag{5.6}$$

[4]) Recall the Lie-algebra isomorphism $\Lambda^2T^*M \cong C^2_u(M) := \{\mathbf{m}_u(a) \mid a \in \Lambda^2T^*M\}$.

to define an operator $\nabla^{\mathbb{A}}\colon \Omega^*(B,\pi_*\mathcal{E})^{\pm} \to \Omega^*(B,\pi_*\mathcal{E})^{\mp}$ by

$$\nabla^{\mathbb{A}} := \lim_{u\to 0} D_{\mathbb{A},u} = \lim_{u\to 0}\big(\mathbf{m}_{\mathrm{u}} \circ \mathbb{A}^{\mathbb{E},u}\big) \tag{5.7}$$

Equivallently we may write $\nabla^{\mathbb{A}} = \mathbf{m}_0 \circ \mathbb{A}^{\mathbb{E},0}$ if we use definition (5.5). Hence $\nabla^{\mathbb{A}}$ can be understood as a kind of 'Dirac operator' on the bundle $\mathbb{E}$ provided with the 'Clifford module'-structure which is defined by the degenerate Clifford action m_0.

Before studying this operator further, recall that $S \in \Gamma(T^*_H M \otimes \operatorname{End} T(M/B)) \cong \Gamma(T^*_H M \otimes T^*(M/B) \otimes T(M/B)$ defined by $S(Z,\theta,X) := \theta(\nabla^{M/B}_Z X - P[Z,X])$ for $Z \in \Gamma(T_H M)$, $\theta \in \Gamma(T^*(M/B)$ and $X \in \Gamma(T(M/B))$ is the second fundamental form associated to a family $\pi\colon M \to B$ of Riemannian manifolds with a given splitting $TM = T(M/B) \oplus T_H M$ and a connection $\nabla^{M/B}$ on the vertical bundle $T(M/B)$, cf. [5]. Here $P\colon TM \to T(M/B)$ denotes the projection map with kernel the choosen horizontal space. Now we obtain the following

Lemma 5.1. *The operator $\nabla^{\mathbb{A}}\colon \Gamma(\pi_*\mathcal{E})^{\pm} \to \Omega^*(B,\pi_*\mathcal{E})^{\mp}$ defined by $\nabla^{\mathbb{A}} := \mathbf{m}_0 \circ \mathbb{A}^{\mathbb{E},0}$ is a superconnection on the direct image $\pi_*\mathcal{E}$ which is explicitly given by*

$$\nabla^{\mathbb{A}} = \mathbf{c} \circ \mathbb{A} + \epsilon \circ (\mathbb{A} + \frac{1}{2}k) + \mathbf{m}_0 \circ \Omega \tag{5.8}$$

where $k \in \Omega^1(M)$ defined by $k(Z) := tr\big(S(Z)\big)$ denotes the mean curvature and $\Omega \in \Gamma(\Lambda^2 T^*_H M \otimes T(M/B))$ is the curvature of the connection $\nabla^{M/B}$ associated to the family $\pi\colon M \to B$ of Riemannian manifolds.

Proof: We follow the proof of Proposition 10.15 of [5]: Obviously, the operator $\nabla^{\mathbb{A}}$ satisfies $\nabla^{\mathbb{A}}(\nu s) = (\epsilon \circ \nabla^B \nu)s + (-1)^{|\,\nu\,|}\nu\nabla^{\mathbb{A}} s$ for all $\nu \in \Omega^*(B)$ and $s \in \Gamma(\mathbb{E})$. Since ∇^B is the Levi-Civita associated to the choosen metric g_B on the base and therefore torsion-free, we see that $\epsilon \circ \nabla^B = d_B$ is the exterior covariant derivative on the base. Thus, $\nabla^{\mathbb{A}}$ is a superconnection on $\pi_*\mathcal{E}$.

For proving the explicit formula (5.8), we observe that the splitting of the cotangent bundle $T^*M = \pi^*T^*B \oplus T^*(M/B)$ implies $\Lambda T^*M = \pi^*\Lambda T^*B \otimes \Lambda T^*(M/B)$. Thus the quantisation map $\mathbf{m}_0\colon \Lambda T^*M \xrightarrow{\cong} C_0(M) := \pi^*\Lambda T^*B \otimes C(M/B)$ associated to the degenerate Clifford structure on $\mathbb{E}$ is given by $\mathbf{m}_0 = \epsilon \otimes \mathbb{1}_{C(M/B)} + \mathbb{1}_{\pi^*\Lambda T^*B} \otimes \mathbf{c}$ when restricted to T^*M. Furthermore, because of definition (5.5) we know

$$\nabla^{\mathbb{A}} = \mathbf{c} \circ \mathbb{A}^{\mathbb{E},\oplus} + \epsilon \circ \mathbb{A}^{\mathbb{E},\oplus} + \frac{1}{2}\, m_0(\omega). \tag{5.9}$$

Using Lemma 10.13 of [5] which tells us $m_0(\omega) = \frac{1}{2}(\epsilon \circ k) + \mathbf{m}_0 \circ \Omega$ completes the proof of equation (5.8).

□

Now assume that the superconnection $\mathbb{A}$ on $\mathcal{E}$ consists only of the connection part $\mathbb{A}_{[1]} = \nabla^{\mathcal{E}}$. Then, by construction, the corresponding superconnection $\nabla\!\!\!\!\nabla^{\nabla^{\mathcal{E}}}$ on $\pi_*\mathcal{E}$ is the Bismut superconnection. Hence we call the operator $\nabla\!\!\!\!\nabla^{\mathbb{A}}$ on $\pi_*\mathcal{E}$ defined by (5.7) 'the generalized Bismut superconnection'. In any case $\nabla\!\!\!\!\nabla^{\mathbb{A}}_{[0]} = \mathsf{D}_{\mathbb{A}}$ holds with $\mathsf{D}_{\mathbb{A}}$ being a family of Dirac operators defined by the superconnection $\mathbb{A}$ as above. Thus, the generalized Bismut superconnection $\nabla\!\!\!\!\nabla^{\mathbb{A}}$ corresponds to an arbitrary family of Dirac operators.

Given any connection $\nabla\colon \Gamma(\mathbb{E}) \to \Gamma(T^*M \otimes \mathbb{E})$ we will call the second order operator $\triangle^{\nabla}_{M/B}$ on $\mathbb{E}$ defined by $\triangle^{\nabla}_{M/B} := g^{ij}_{M/B}\big(\nabla_i\nabla_j - \nabla_{\nabla^{M/B}_i e_j}\big)$ the 'vertical' connection laplacian associated to ∇. Note that we adopt the convention that the indices $i,\ j,\ \ldots$ label vertical vectors. Using this notation, we finally state the analogue of Theorem 4.2 which provides the formula for supercurvature $\big(\nabla\!\!\!\!\nabla^{\mathbb{A}}\big)^2$ of the generalized Bismut superconnection:

Theorem 5.2. *Let $\nabla\!\!\!\!\nabla^{\mathbb{A}}\colon \Gamma(\pi_*\mathcal{E})^{\pm} \to \Omega^*(B, \pi_*\mathcal{E})$ be the generalized Bismut superconnection corresponding to the Clifford superconnection $\mathbb{A}$ on the Clifford module $\mathcal{E}$ over $C(M/B)$. Then*

$$\big(\nabla\!\!\!\!\nabla^{\mathbb{A}}\big)^2 = \triangle^{\hat{\nabla}}_{M/B} + \frac{r_{M/B}}{4} + \mathbf{m}_0\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big) + \mathbf{m}_0\big(\bar{\mathbb{A}}\big)^2 - \mathbf{m}_0\big(\bar{\mathbb{A}}^2\big) + ev_{g_{M/B}}\big(\beta_0(\bar{\mathbb{A}}) \cdot \beta_0(\bar{\mathbb{A}})\big)$$

where $r_{M/B}$ denotes the scalar curvature of $\nabla^{M/B}$, $\hat{\nabla} := \mathbb{A}_{[1]} + \beta_0(\bar{\mathbb{A}})$ determines the 'vertical' connection laplacian $\triangle^{\hat{\nabla}}_{M/B}$, the form $\mathbb{F}(\mathbb{A})^{\mathcal{E}/S} \in \Omega^*(M, \mathrm{End}_{C(M/B)}\mathcal{E})$ is the twisting curvature of $\mathbb{A}$ and $\beta_0(\bar{\mathbb{A}}) \in \Omega^1(M, \mathrm{End}\ \mathbb{E})$ is defined by $\beta_0(\bar{\mathbb{A}}) := dx^j \otimes \mathbf{m}_0\big(i(\partial_j)\bar{\mathbb{A}}\big)$ with respect to a local vertical coordinate frame.

The proof of this theorem is similar to our proof of the generalized Lichnerowicz formula and can be found in [1]. However, in order to do so, the extension of all the previous results in section 3 to this case is indispensable. For instance, the analogue $\mathbf{g_{M/B}}$ of the canonical projection map $\mathbf{g}$, cf. (3.2), can be defined by

$$\mathbf{g_{M/B}}\colon \Omega^*(M, \mathrm{End}\ \mathbb{E}) \xrightarrow{\mathbf{m_0}} \Omega^0(M, \mathrm{End}\ \mathbb{E}) \overset{\mu(\gamma_{M/B})}{\longrightarrow} \Omega^1(M, \mathrm{End}\ \mathbb{E}). \tag{5.10}$$

Recall that here $\mathbf{m_0}$ denotes the composition of the 'quantisation map' $\mathbf{m_0}$ and the action $m_0\colon \pi^*\Lambda T^*B \hat{\otimes} C(M/B) \to \mathrm{End}\ \mathbb{E}$ by abuse of notation, and the one-form

$\gamma_{M/B}$ is defined by $\gamma_{M/B} = -\frac{1}{n}(g_{M/B})_{ij} dx^i \otimes c(dx^j) \in \Omega^1(M, \text{End } \mathbb{E})$ where $n := \dim M_z,\ z \in B$ is the fibre dimension. The End $\mathbb{E}$-vallued one forms decompose into components $\Omega^1(M, \text{End } \mathbb{E}) \cong \Gamma(\pi^* T^* B \otimes \text{End } \mathbb{E}) \oplus \Gamma(T^*(M/B) \otimes \text{End } \mathbb{E})$ and we calculate $m_0 \circ \mathbf{g_{M/B}} = (\epsilon \otimes 1\!\!1_{\mathbb{E}} + c \otimes 1\!\!1_{\mathbb{E}}) \circ \mu(\gamma_{M/B}) \circ \mathbf{m_0} = c \circ \mu(\gamma_{M/B}) \circ \mathbf{m_0} = \mathbf{m_0}$ because $\gamma_{M/B}$ is vertical [5]). In turn this implies an analogue of Lemma 4.1 above. Thus, generalized Bismut superconnections can be understood as 'quantisation' (with respect to the 'degenerate' quantisation map $\mathbf{m}_0$) of the theory of connections on the 'degenerate Clifford module' $\mathbb{E}$. This is an extension of Quillen's principle as mentioned after Lemma 4.1 concerning the relation of Dirac operators and connections to the setting of families of Dirac operators. For more details and a complete proof of Theorem 5.2 we refer once more to [1] where it is also shown how to compute the Chern character of a generalized Bismut superconnection using this formula.

6. Conclusion

We have studied Dirac operators acting on sections of a Clifford module $\mathcal{E}$ over a Riemannian manifold M. Motivated by the fact that any Clifford superconnection $\mathbb{A}$ on $\mathcal{E}$ uniquely determines a Dirac operator $D_{\mathbb{A}}$, in this paper we have emphasized the supersymmeric approach using Quillen's super-formalism. We have proven the supersymmetric version (4.3) of the decomposition formula for the square of a Dirac operator $D_{\mathbb{A}}$ which generalizes the classical result [8] due to Lichnerowicz. Associated to a family of (arbitrary) Dirac operators $\mathsf{D}_{\mathbb{A}} := \{D^z_{\mathbb{A}} \mid z \in B\}$ parametrized by a not necessarily finite dimensional manifold B we have defined the notion of 'generalized Bismut superconnection' $\nabla^{\mathbb{A}}$. This generalizes Bismut's construction [4]. Similarly we have obtained a simple formula for its supercurvature $\left(\nabla^{\mathbb{A}}\right)^2$, extending the generalized Lichnerowicz formula (4.3). This might be seen as a first step to prove the local Atiyah-Singer index theorem also for families of arbitrary Dirac operators [1]. For applications of the generalized Lichnerowicz formula in physics, we refer to [2] and [3].

5) Recall, that $c\colon C(M/B) \to \text{End } \mathcal{E}$ denotes the vertical Clifford action.

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